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ON THE EIGENVALUES OF STOCHASTIC MATRICES

by M. DA GRAÇA MARQUES[6] and G. N. DE OLIVEIRA[7]

Let $A = [a_{ij}]$ be an $n \times n$ matrix with nonnegative entries. A is said to be ω-stochastic if, for every i, $\sum_{j=1}^{n} a_{ij} = \omega$. When $\omega = 1$ we call it stochastic.

In 1949 Suleimanova [4] posed the following problem. Given n complex numbers $\lambda_1, \dots, \lambda_n$, find a necessary and sufficient condition for the existence of an $n \times n$ stochastic matrix with $\lambda_1, \dots, \lambda_n$ as eigenvalues. Some partial results on this question are known today, but the problem remains unsolved for $n > 3$. In particular the knowledge of sufficient conditions is very poor when not all of the λ_i's are real. The so called $\mathscr{L}$-transformation developed in [3] seems to be a useful tool for tackling this problem.

Let A be an $n \times n$ matrix (not necessarily stochastic), and let C, X, T be $k \times k$, $n \times k$, $(n+k)\times(n+k)$ matrices respectively with T invertible. We

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define $\mathscr{L}_T^{\binom{X}{C}}(A)$ by

$$\mathscr{L}_T^{\binom{X}{C}}(A)=T\begin{bmatrix} A & X \\ 0 & C \end{bmatrix}T^{-1}.$$

It is clear that the eigenvalues of $\mathscr{L}_T^{\binom{X}{C}}(A)$ are those of A together with those of C.

Since we do not have space for details, we show with an example how this transformation can be used for constructing ω-stochastic matrices with prescribed eigenvalues. Let

$$A=\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix},$$

which is 5-stochastic with eigenvalues 5 and 2. We construct now a 5-stochastic matrix with eigenvalues 5, 2, $1+i$, $1-i$. Let

$$C=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

The eigenvalues of C are $1\pm i$. Now take

$$X=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad T=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We have

$$\mathscr{L}_T^{\binom{X}{C}}(A)=\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix},$$

which is a 5-stochastic matrix with the required eigenvalues.

Of course, in proving theorems with the $\mathcal{L}$-transformation we can iterate, and the crucial problem in each step is the choice of X and T. Notice that in the above example, we took as T a matrix which can be regarded as obtained from the identity matrix by adding certain rows to other rows. It is easy to invert matrices of this type.

Also, of course, the $\mathcal{L}$-transformation can be combined with other results in order to obtain new sufficient conditions. As an example we present Theorem 3 below. For the proof of Theorem 3 we need the following theorems of M. Fiedler.

THEOREM 1 [1]. *Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, $a_1 \geqslant \cdots \geqslant a_n$ ($\geqslant 0$) satisfy*

$$\sum_{i=1}^{s} \lambda_i \geqslant \sum_{i=1}^{s} a_i, \qquad s = 1,\dots,n-1,$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i,$$

$$\lambda_k \leqslant a_{k-1}, \qquad k = 2,\dots,n-1.$$

Then there exists a nonnegative symmetric matrix with eigenvalues $\lambda_1,\dots,\lambda_n$ and diagonal entries $a_1,\dots,a_n$.

THEOREM 2 [2]. *Let $\lambda_1,\dots,\lambda_n$, $a_1,\dots,a_n$ satisfy*

$$a_i \geqslant 0, \qquad i = 1,\dots,n,$$

$$a_1 = \max a_i,$$

$$\lambda_j \leqslant a_j, \qquad j = 2,\dots,n,$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i,$$

Then there exists a nonnegative symmetric matrix with eigenvalues $\lambda_1,\dots,\lambda_n$ and diagonal entries $a_1,\dots,a_n$.

THEOREM 3. *Let $\lambda_1,\dots,\lambda_n$ be real numbers and $z_1,\dots,z_k$ complex numbers. Assume $a_1,\dots,a_n$ is a set of real numbers satisfying the conditions* *of either Theorem 1 or Theorem 2. Let $z_j = \alpha_j + i\beta_j$. If there is a permutation σ of $\{1,\dots,n\}$ such that*

(1) *if* $\alpha_j \geqslant 0$, $2|\beta_j| + \alpha_j \leqslant a_{\sigma(j)}$,
(2) *if* $\alpha_j < 0$, $2\max\{|\alpha_j|, |\beta_j|\} \leqslant a_{\sigma(j)}$,

then there is an $(n+2k)$-square λ_1-stochastic matrix with eigenvalues $\lambda_1,\dots,\lambda_n$, $z_1,\dots,z_k$, $\bar{z}_1,\dots,\bar{z}_k$.

Sketch of the proof. By Theorem 1 or 2 there is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1,\dots,\lambda_n$ and diagonal elements $a_1,\dots,a_n$. This implies the existence of an $n \times n$ λ_1-stochastic matrix with the same eigenvalues and diagonal elements. Call it A. Now starting with A and using successively the $\mathscr{L}$-transformation making use of conditions (1) and (2), we can construct an $(n+2k)$-square λ_1-stochastic matrix with the required eigenvalues.

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PARALLEL TRIANGULARIZATION OF A SPARSE MATRIX ON A DISTRIBUTED-MEMORY MULTIPROCESSOR USING FAST GIVENS ROTATIONS

by J. DUATO[8]

1. *Introduction*

Many applications require the solution of a least-squares (LS) problem from a coefficient matrix and a measurement vector. In some cases the solution must be obtained within a short period of time, requiring great

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